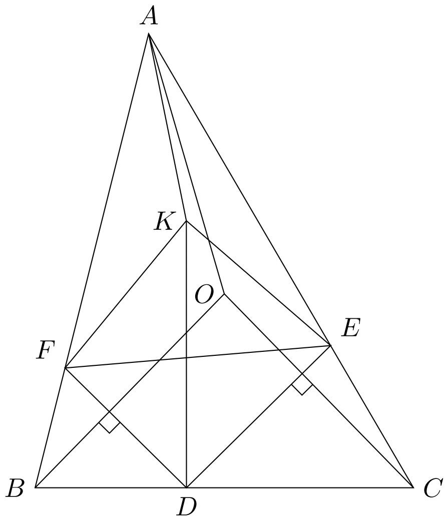
European Girls' Mathematical Olympiad 2012—Day 1 Solutions

Problem 1. Let ABC be a triangle with circumcentre O. The points D, E and F lie in the interiors of the sides BC, CA and AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By *interior* we mean, for example, that the point D lies on the line BC and D is between B and C on that line.)

Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.

Origin. Netherlands (Merlijn Staps).



Solution 1 (submitter). Let ℓ_C be the tangent at C to the circumcircle of $\triangle ABC$. As $CO \perp \ell_C$, the lines DE and ℓ_C are parallel. Now we find that

$$\angle CDE = \angle(BC, \ell_C) = \angle BAC,$$

hence the quadrilateral $BDEA$ is cyclic. Analogously, we find that the quadrilateral $CDFA$ is cyclic. As we now have $\angle CDE = \angle A = \angle FDB$, we conclude that the line BC is the external angle bisector of $\angle EDF$. Furthermore, $\angle EDF = 180° - 2\angle A$. Since K is the circumcentre of $\triangle AEF$, $\angle FKE = 2\angle FAE = 2\angle A$. So $\angle FKE + \angle EDF = 180°$, hence K lies on the circumcircle of $\triangle DEF$. As $|KE| = |KF|$, we have that K is the midpoint of the arc EF of this circumcircle. It is well known that this point lies on the internal angle bisector of $\angle EDF$. We conclude that DK is the internal angle bisector of $\angle EDF$. Together with the fact that BC is the external angle bisector of $\angle EDF$, this yields that $DK \perp BC$, as desired.

Solution 2 (submitter). As in the previous solution, we show that the quadrilaterals $BDEA$ and $CDFA$ are both cyclic. Denote by M and L respectively the circumcentres of these quadrilaterals. We will show that the quadrilateral $KLOM$ is a parallelogram. The lines KL and MO are the perpendicular bisectors of the line segments AF and AB, respectively. Hence both KL and MO are perpendicular to AB, which yields $KL \parallel MO$. In the same way we can show that the lines KM and LO are both perpendicular to AC and hence parallel as well. We conclude that $KLOM$ is indeed a parallelogram. Now, let K', L', O' and M' be the respective projections of K, L, O and M to BC. We have to show that $K' = D$. As L lies on the perpendicular bisector of CD, we have that L' is the midpoint of CD. Similarly, M' is the midpoint of BD and O' is the midpoint of BC. Now we are going to use directed lengths. Since $KLOM$ is a parallelogram, $M'K' = O'L'$. As

$$O'L' = O'C - L'C = \tfrac{1}{2} \cdot (BC - DC) = \tfrac{1}{2} \cdot BD = M'D,$$

we find that $M'K' = M'D$, hence $K' = D$, as desired.

Solution 3 (submitter). Denote by ℓ_A, ℓ_B and ℓ_C the tangents at A, B and C to the circumcircle of $\triangle ABC$. Let A' be the point of intersection of ℓ_B and ℓ_C and define B' and C' analogously. As in the first solution, we find that $DE \parallel \ell_C$ and $DF \parallel \ell_B$. Now, let Q be the point of intersection of DE and ℓ_A and let R be the point of intersection of DF and ℓ_A. We easily find $\triangle AQE \sim \triangle AB'C$. As $|B'A| = |B'C|$, we must have $|QA| = |QE|$, hence $\triangle AQE$ is isosceles. Therefore the perpendicular bisector of AE is the internal angle bisector of $\angle EQA = \angle DQR$. Analogously, the perpendicular bisector of AF is the internal angle bisector of $\angle DRQ$. We conclude that K is the incentre of $\triangle DQR$, thus DK is the angle bisector of $\angle QDR$. Because the sides of the triangles $\triangle QDR$ and $\triangle B'A'C'$ are pairwise parallel, the angle bisector DK of $\angle QDR$ is parallel to the angle bisector of $\angle B'A'C'$. Finally, as the angle bisector of $\angle B'A'C'$ is easily seen to be perpendicular to BC (as it is the perpendicular bisector of this segment), we find that $DK \perp BC$, as desired.

Remark (submitter). The fact that the quadrilateral $BDEA$ is cyclic (which is an essential part of the first two solutions) can be proven in various ways. Another possibility is as follows. Let P be the midpoint of BC. Then, as $\angle CPO = 90°$, we have $\angle POC = 90° - \angle OCP$. Let X be the point of intersection of DE and CO, then we have that $\angle CDE = \angle CDX = 90° - \angle XCD = 90° - \angle OCP$. Hence $\angle CDE = \angle POC = \frac{1}{2}\angle BOC = \angle BAC$. From this we can conclude that $BDEA$ is cyclic.

Solution 4 (PSC). This is a simplified variant of Solution 1. $\angle COB = 2\angle A$ (angle at centre of circle ABC) and $OB = OC$ so $\angle OBC = \angle BCO = 90° - \angle A$. Likewise $\angle EKF = 2\angle A$ and $\angle KFE = \angle FEK = 90° - \angle A$. Now because $DE \perp CO$, $\angle EDC = 90° - \angle DCO = 90° - \angle BCO = \angle A$ and similarly $\angle BDF = \angle A$, so $\angle FDE = 180° - 2\angle A$. So quadrilateral $KFDE$ is cyclic (opposite angles), so (same segment) $\angle KDE = \angle KFE = 90° - \angle A$, so $\angle KDC = 90°$ and DK is perpendicular to BC.

Problem 2. Let n be a positive integer. Find the greatest possible integer m, in terms of n, with the following property: a table with m rows and n columns can be filled with real numbers in such a manner that for any two different rows $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ the following holds:

$$\max(|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|) = 1.$$

Origin. Poland (Tomasz Kobos).

Solution 1 (submitter). The largest possible m is equal to 2^n.

In order to see that the value 2^n can be indeed achieved, consider all binary vectors of length n as rows of the table. We now proceed with proving that this is the maximum value.

Let $[a_k^i]$ be a feasible table, where $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Let us define undirected graphs G_1, G_2, $\ldots$, G_n, each with vertex set $\{1, 2, \ldots, m\}$, where $ij \in E(G_k)$ if and only if $|a_k^i - a_k^j| = 1$ (by $E(G_k)$ we denote the edge set of the graph G_k). Observe the following two properties.

(1) Each graph G_k is bipartite. Indeed, if it contained a cycle of odd length, then the sum of ± 1 along this cycle would need to be equal to 0, which contradicts the length of the cycle being odd.

(2) For every $i \neq j$, $ij \in E(G_k)$ for some k. This follows directly from the problem statement.

For every graph G_k fix some bipartition (A_k, B_k) of $\{1, 2, \ldots, m\}$, i.e., a partition of $\{1, 2, \ldots, m\}$ into two disjoint sets A_k, B_k such that the edges of G_k traverse only between A_k and B_k. If $m > 2^n$, then there are two distinct indices i, j such that they belong to exactly the same parts A_k, B_k, that is, $i \in A_k$ if and only if $j \in A_k$ for all $k = 1, 2, \ldots, n$. However, this means that the edge ij cannot be present in any of the graphs G_1, G_2, $\ldots$, G_n, which contradicts (2). Therefore, $m \leq 2^n$.

Solution 2 (PSC). In any table with the given property, the least and greatest values in a column cannot differ by more than 1. Thus, if each value that is neither least nor greatest in its column is changed to be equal to either the least or the greatest value in its column (arbitrarily), this does not affect any $|a_i - b_i| = 1$, nor does it increase any difference above 1, so the table still has that given property. But after such a change, for any choice of what the least and greatest values in each column are, there are only two possible choices for each entry in the table (either the least or the greatest value in its column); that is, only 2^n possible distinct rows, and the given property implies that all rows must be distinct. As in the previous solution, we see that this number can be achieved.

Solution 3 (Coordinators). We prove by induction on n that $m \leq 2^n$.

First suppose $n = 1$. If real numbers x and y have $|x - y| = 1$ then $\lfloor x \rfloor$ and $\lfloor y \rfloor$ have opposite parities and hence it is impossible to find three real numbers with all differences 1. Thus $m \leq 2$.

Suppose instead $n > 1$. Let a be the smallest number appearing in the first column of the table; then every entry in the first column of the table lies in the interval $[a, a+1]$. Let A be the collection of rows with first entry a and B be the collection of rows with first entry in $(a, a+1]$. No two rows in A differ by 1 in their first entries, so if we list the rows in A and delete their first entries we obtain a table satisfying the conditions of the problem with n replaced by $n-1$; thus, by the induction hypothesis, there are at most 2^{n-1} rows in A. Similarly, there are at most 2^{n-1} rows in B. Hence $m \leq 2^{n-1} + 2^{n-1} = 2^n$. As before, this number can be achieved.

Solution 4 (Coordinators). Consider the rows of the table as points of $\mathbb{R}^n$. As the values in each column differ by at most 1, these points must lie in some n-dimensional unit cube C. Consider the unit cubes centred on each of the m points. The conditions of the problem imply that the interiors of these unit cubes are pairwise disjoint. But now C has volume 1, and each of these cubes intersects C in volume at least 2^{-n}: indeed, if the unit cube centred on a point of C is divided into 2^n cubes of equal size then one of these cubes must lie entirely within C. Hence $m \leq 2^n$. As before, this number can be achieved.

Solution 5 (Coordinators). Again consider the rows of the table as points of $\mathbb{R}^n$. The conditions of the problem imply that these points must all lie in some n-dimensional unit cube C, but no two of the points lie in any smaller cube. Thus if C is divided into 2^n equally-sized subcubes, each of these subcubes contains at most one row of the table, giving $m \leq 2^n$. As before, this number can be achieved.

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(\, yf(x+y) + f(x) \,\big) = 4x + 2yf(x+y)$$

for all $x, y \in \mathbb{R}$.

Origin. Netherlands (Birgit van Dalen).

Solution 1 (submitter). Setting $y = 0$ yields

$$f(f(x)) = 4x, \tag{1}$$

from which we derive that f is a bijective function. Also, we find that

$$f(0) = f(4 \cdot 0) = f(f(f(0))) = 4f(0),$$

hence $f(0) = 0$. Now set $x = 0$ and $y = 1$ in the given equation and use (1) again:

$$4 = f(f(1)) = 2f(1),$$

so $f(1) = 2$ and therefore also $f(2) = f(f(1)) = 4$. Finally substitute $y = 1 - x$ in the equation:

$$f(2(1-x) + f(x)) = 4x + 4(1-x) = 4 = f(2) \quad \text{for all } x \in \mathbb{R}.$$

As f is injective, from this it follows that $f(x) = 2 - 2(1-x) = 2x$. It is easy to see that this function satisfies the original equation. Hence the only solution is the function defined by $f(x) = 2x$ for all $x \in \mathbb{R}$.

Solution 2 (Coordinators). Setting $y = 0$ in the equation we see

$$f(f(x)) = 4x$$

so f is a bijection. Let $\kappa = f^{-1}(2)$ and set $x + y = \kappa$ in the original equation to see

$$f(2\kappa - 2x + f(x)) = 4\kappa.$$

As the right hand side is independent of x and f is injective, $2\kappa - 2x + f(x)$ is constant, i.e. $f(x) = 2x + \alpha$.

Substituting this into the original equation, we see that $2x + \alpha$ is a solution to the original equation if and only if $4(y^2 + xy + x) + (3 + 2y)\alpha = 4(y^2 + xy + x) + 2y\alpha$ for all x, y, i.e. if and only if $\alpha = 0$. Thus the unique solution to the equation is $f(x) = 2x$.

Problem 4. A set A of integers is called *sum-full* if $A \subseteq A + A$, i.e. each element $a \in A$ is the sum of some pair of (not necessarily different) elements $b, c \in A$. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of A.

Does there exist a sum-full zero-sum-free set of integers?

Origin. Romania (Dan Schwarz).

Remark. The original formulation of this problem had a weaker definition of *zero-sum-free* that did not require all nonzero integers to be sums of finite nonempty subsets of A.

Solution (submitter, adapted). The set $A = \{F_{2n} : n = 1, 2, \ldots\} \cup \{-F_{2n+1} : n = 1, 2, \ldots\}$, where F_k is the k^{th} Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_{k+2} = F_{k+1} + F_k$ for $k \geq 1$) qualifies for an example. We then have $F_{2n} = F_{2n+2} + (-F_{2n+1})$ and $-F_{2n+1} = (-F_{2n+3}) + F_{2n+2}$ for all $n \geq 1$, so A is sum-full (and even with unique representations). On the other hand, we can never have

$$0 = \sum_{i=1}^{s} F_{2n_i} - \sum_{j=1}^{t} F_{2n_j+1},$$

owing to the fact that Zeckendorf representations are known to be unique.

It remains to be shown that all nonzero values can be represented as sums of distinct numbers 1, -2, 3, -5, 8, -13, 21, ... This may be done using a greedy algorithm: when representing n, the number largest in magnitude that is used is the element $m = \pm F_k$ of A that is closest to 0 subject to having the same sign as n and $|m| \geq |n|$. That this algorithm terminates without using any member of A twice is a straightforward induction on k; the base case is $k = 2$ ($m = 1$) and the induction hypothesis is that for all n for which the above algorithm starts with $\pm F_\ell$ with $\ell \leq k$, it terminates without having used any member of A twice and without having used any $\pm F_j$ with $j > \ell$.

Remark (James Aaronson and Adam P Goucher). Let n be a positive integer, and write $u = 2^n$; we claim that the set

$$\{1, 2, 4, \ldots, 2^{n-1}, -u, u + 1, -(2u + 1), 3u + 2, -(5u + 3), 8u + 5, \ldots\}$$

is a sum-full zero-sum-free set. The proof is similar to that used for the standard examples.